

# Characterizations of some classes of finite $\sigma$ -soluble $P\sigma T$ -groups

–Dedicated to Professor J.C. Beidleman on the occasion of his 80-th birthday

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## Abstract

Let  $\sigma = \{\sigma_i | i \in I\}$  be some partition of the set of all primes  $\mathbb{P}$  and  $G$  a finite group.  $G$  is said to be  $\sigma$ -soluble if every chief factor  $H/K$  of  $G$  is a  $\sigma_i$ -group for some  $i = i(H/K)$ .

A set  $\mathcal{H}$  of subgroups of  $G$  is said to be a *complete Hall  $\sigma$ -set* of  $G$  if every member  $\neq 1$  of  $\mathcal{H}$  is a Hall  $\sigma_i$ -subgroup of  $G$  for some  $\sigma_i \in \sigma$  and  $\mathcal{H}$  contains exactly one Hall  $\sigma_i$ -subgroup of  $G$  for every  $i \in I$  such that  $\sigma_i \cap \pi(G) \neq \emptyset$ . A subgroup  $A$  of  $G$  is said to be  $\sigma$ -permutable in  $G$  if  $G$  has a complete Hall  $\sigma$ -set  $\mathcal{H}$  such that  $AH^x = H^x A$  for all  $x \in G$  and all  $H \in \mathcal{H}$ .

We obtain characterizations of finite  $\sigma$ -soluble groups  $G$  in which  $\sigma$ -permutability is a transitive relation in  $G$ .

## 1 Introduction

Throughout this paper, all groups are finite and  $G$  always denotes a finite group. Moreover,  $\mathbb{P}$  is the set of all primes,  $\pi = \{p_1, \dots, p_n\} \subseteq \mathbb{P}$  and  $\pi' = \mathbb{P} \setminus \pi$ . If  $n$  is an integer, the symbol  $\pi(n)$  denotes the set of all primes dividing  $n$ ; as usual,  $\pi(G) = \pi(|G|)$ , the set of all primes dividing the order of  $G$ .  $G$  is said to be a  $D_\pi$ -group if  $G$  possesses a Hall  $\pi$ -subgroup  $E$  and every  $\pi$ -subgroup of  $G$  is contained in some conjugate of  $E$ .

In what follows,  $\sigma$  is some partition of  $\mathbb{P}$ , that is,  $\sigma = \{\sigma_i | i \in I\}$ , where  $\mathbb{P} = \cup_{i \in I} \sigma_i$  and  $\sigma_i \cap \sigma_j = \emptyset$  for all  $i \neq j$ ;  $\Pi$  is always supposed to be a subset of the set  $\sigma$  and  $\Pi' = \sigma \setminus \Pi$ .

By the analogy with the notation  $\pi(n)$ , we write  $\sigma(n)$  to denote the set  $\{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ ;  $\sigma(G) = \sigma(|G|)$ .  $G$  is said to be:  $\sigma$ -primary [1] if  $G$  is a  $\sigma_i$ -group for some  $i$ ;  $\sigma$ -decomposable [2] or  $\sigma$ -nilpotent [3] if  $G = G_1 \times \dots \times G_n$  for some  $\sigma$ -primary groups  $G_1, \dots, G_n$ ;  $\sigma$ -soluble [1] if every chief factor of  $G$  is  $\sigma$ -primary; a  $\sigma$ -full group of Sylow type [1] if every subgroup  $E$  of  $G$  is a  $D_{\sigma_i}$ -group for every  $\sigma_i \in \sigma(E)$ . Note in passing, that every  $\sigma$ -soluble group is a  $\sigma$ -full group of Sylow type [4].

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A set  $\mathcal{H}$  of subgroups of  $G$  is a *complete Hall  $\sigma$ -set* of  $G$  [4, 5] if every member  $\neq 1$  of  $\mathcal{H}$  is a Hall  $\sigma_i$ -subgroup of  $G$  for some  $\sigma_i \in \sigma$  and  $\mathcal{H}$  contains exactly one Hall  $\sigma_i$ -subgroup of  $G$  for every  $\sigma_i \in \sigma(G)$ .

Recall also that a subgroup  $A$  of  $G$  is said to be  *$\sigma$ -subnormal* in  $G$  [1] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either  $A_{i-1} \trianglelefteq A_i$  or  $A_i/(A_{i-1})_{A_i}$  is  $\sigma$ -primary for all  $i = 1, \dots, n$ .

**Definition 1.1.** We say that a subgroup  $A$  of  $G$  is said to be  *$\sigma$ -quasinormal* or  *$\sigma$ -permutable* in  $G$  if  $G$  possesses a complete Hall  $\sigma$ -set and  $A$  permutes with each Hall  $\sigma_i$ -subgroup  $H$  of  $G$ , that is,  $AH = HA$  for all  $i \in I$ .

**Remark 1.2.** Using Theorem B in [1], it is not difficult to show that if  $G$  possesses a complete Hall  $\sigma$ -set  $\mathcal{H}$  such that  $AH^x = H^x A$  for all  $H \in \mathcal{H}$  and all  $x \in G$ , then  $A$  is  $\sigma$ -permutable in  $G$ .

**Remark 1.3.** (i) In the classical case when  $\sigma = \sigma^0 = \{\{2\}, \{3\}, \dots\}$ :  $G$  is  $\sigma^0$ -soluble (respectively  $\sigma^0$ -nilpotent) if and only if  $G$  is soluble (respectively nilpotent);  $\sigma^0$ -permutable subgroups are also called  *$S$ -quasinormal* or  *$S$ -permutable* [6, 7]. A subgroup  $A$  of  $G$  is subnormal in  $G$  if and only if it is  $\sigma^0$ -subnormal in  $G$ .

(ii) In the other classical case when  $\sigma = \sigma^\pi = \{\pi, \pi'\}$ :  $G$  is  $\sigma^\pi$ -soluble (respectively  $\sigma^\pi$ -nilpotent) if and only if  $G$  is  $\pi$ -separable (respectively  $\pi$ -decomposable, that is,  $G = O_\pi(G) \times O_{\pi'}(G)$ ); a subgroup  $A$  of a  $\pi$ -separable group  $G$  is  $\sigma^\pi$ -permutable in  $G$  if and only if  $A$  permutes with all Hall  $\pi$ -subgroups and with all Hall  $\pi'$ -subgroups of  $G$ . A subgroup  $A$  of a  $\pi$ -separable group  $G$  is  $\sigma^\pi$ -subnormal in  $G$  if and only if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that  $A_i/(A_{i-1})_{A_i}$  is either a  $\pi$ -group or a  $\pi'$ -group for all  $i = 1, \dots, n$ .

(iii) In fact, in the theory of  $\pi$ -soluble groups ( $\pi = \{p_1, \dots, p_n\}$ ) we deal with the partition  $\sigma = \sigma^{0\pi} = \{\{p_1\}, \dots, \{p_n\}, \pi'\}$  of  $\mathbb{P}$ . Note that  $G$  is  $\sigma^{0\pi}$ -soluble (respectively  $\sigma^{0\pi}$ -nilpotent) if and only if  $G$  is  $\pi$ -soluble (respectively  $G = O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)$ ). A subgroup  $A$  of a  $\pi$ -soluble group  $G$  is  $\sigma^{0\pi}$ -permutable in  $G$  if and only if  $A$  permutes with all Hall  $\pi'$ -subgroups and with all Sylow  $p$ -subgroups of  $G$  for all  $p \in \pi$ . Note also that a subgroup  $A$  of  $G$  is  $\sigma^{0\pi}$ -subnormal in  $G$  if and only if it is  $\mathfrak{F}$ -subnormal in  $G$  in the sense of Kegel [8], where  $\mathfrak{F}$  is the class of all  $\pi'$ -groups.

We say that  $G$  is a  *$P\sigma T$ -group* [1] if  $\sigma$ -permutability is a transitive relation in  $G$ , that is, if  $K$  is a  $\sigma$ -permutable subgroup of  $H$  and  $H$  is a  $\sigma$ -permutable subgroup of  $G$ , then  $K$  is a  $\sigma$ -permutable subgroup of  $G$ . In the case when  $\sigma = \sigma^0$ , a  *$P\sigma T$ -group* is also called a  *$PST$ -group* [6]. Note that if  $G = (Q_8 \rtimes C_3) \rtimes (C_7 \rtimes C_3)$  (see [9, p. 50]), where  $Q_8 \rtimes C_3 = SL(2, 3)$  and  $C_7 \rtimes C_3$  is a non-abelian group of order 21, then  $G$  is not a  *$PST$ -group* but  $G$  is a  *$P\sigma T$ -group*, where  $\sigma = \{\{2, 3\}, \{2, 3\}'\}$ .

The description of  *$PST$ -groups* was first obtained by Agrawal [10], for the soluble case, and by Robinson in [11], for the general case. In the further publications, authors (see, for example, the

recent papers [12]–[22] and Chapter 2 in [6]) have found out and described many other interesting characterizations of soluble  $PST$ -groups.

The purpose of this paper is to study  $\sigma$ -soluble  $P\sigma T$ -groups in the most general case (i.e., without any restrictions on  $\sigma$ ). In view of Theorem B in [1],  $G$  is a  $P\sigma T$ -group if and only if every  $\sigma$ -subnormal subgroup of  $G$  is  $\sigma$ -permutable. Being based on this result, here we prove the following revised version of Theorem A in [1].

**Theorem A.** *Let  $D = G^{\mathfrak{N}_\sigma}$ . If  $G$  is a  $\sigma$ -soluble  $P\sigma T$ -group, then the following conditions hold:*

- (i)  $G = D \rtimes M$ , where  $D$  is an abelian Hall subgroup of  $G$  of odd order,  $M$  is  $\sigma$ -nilpotent and every element of  $G$  induces a power automorphism in  $D$ ;
- (ii)  $O_{\sigma_i}(D)$  has a normal complement in a Hall  $\sigma_i$ -subgroup of  $G$  for all  $i$ .

*Conversely, if Conditions (i) and (ii) hold for some subgroups  $D$  and  $M$  of  $G$ , then  $G$  is a  $P\sigma T$ -group.*

In this theorem,  $G^{\mathfrak{N}_\sigma}$  denotes the  $\sigma$ -nilpotent residual of  $G$ , that is, the intersection of all normal subgroups  $N$  of  $G$  with  $\sigma$ -nilpotent quotient  $G/N$ .

**Corollary 1.4.** *If  $G$  is a  $\sigma$ -soluble  $P\sigma T$ -group, then every quotient and every subgroup of  $G$  are  $P\sigma T$ -groups.*

In the case when  $\sigma = \sigma^0$ , we get from Theorem A the following

**Corollary 1.5** (Agrawal [10, Theorem 2.3]). *Let  $D = G^{\mathfrak{N}}$  be the nilpotent residual of  $G$ . If  $G$  is a soluble  $PST$ -group, then  $D$  is an abelian Hall subgroup of  $G$  of odd order and every element of  $G$  induces a power automorphism in  $D$ .*

In the case when  $\sigma = \sigma^\pi$ , we get from Theorem A the following

**Corollary 1.6.**  *$G$  is a  $\pi$ -separable  $P\sigma^\pi T$ -group if and only if the following conditions hold:*

- (i)  $G = D \rtimes M$ , where  $D$  is an abelian Hall subgroup of  $G$  of odd order,  $M$  is  $\pi$ -decomposable and every element of  $G$  induces a power automorphism in  $D$ ;
- (ii)  $O_\pi(D)$  has a normal complement in a Hall  $\pi$ -subgroup of  $G$ ;
- (iii)  $O_{\pi'}(D)$  has a normal complement in a Hall  $\pi'$ -subgroup of  $G$ .

In the case when  $\sigma = \sigma^{0\pi}$ , we get from Theorem A the following

**Corollary 1.7.**  *$G$  is a  $\pi$ -soluble  $P\sigma^{0\pi} T$ -group if and only if the following conditions hold:*

- (i)  $G = D \rtimes M$ , where  $D$  is an abelian Hall subgroup of  $G$  of odd order,  $M = O_{p_1}(M) \times \cdots \times O_{p_n}(M) \times O_{\pi'}(M)$  and every element of  $G$  induces a power automorphism in  $D$ ;
- (ii)  $O_{\pi'}(D)$  has a normal complement in a Hall  $\pi'$ -subgroup of  $G$ .

A natural number  $n$  is said to be a  $\Pi$ -number if  $\sigma(n) \subseteq \Pi$ . A subgroup  $A$  of  $G$  is said to be: a Hall  $\Pi$ -subgroup of  $G$  [1] if  $|A|$  is a  $\Pi$ -number and  $|G : A|$  is a  $\Pi'$ -number; a  $\sigma$ -Hall subgroup of  $G$  if  $A$  is a Hall  $\Pi$ -subgroup of  $G$  for some  $\Pi \subseteq \sigma$ .

The proof of Theorem A is based on many results and observations of the paper [1]. We use also the remarkable result of the paper by Alexandre, Ballester-Bolinches and Pedraza-Aguilera [14] (see also Theorem 2.1.8 in [6]) that in a soluble  $PST$ -group  $G$  any two isomorphic chief factors are  $G$ -isomorphic. Finally, in the proof of Theorem A, the following fact is useful, which is possibly independently interesting.

**Theorem B.** *Let  $G$  have a normal  $\sigma$ -Hall subgroup  $D$  such that: (i)  $G/D$  is a  $P\sigma T$ -group, and (ii) every  $\sigma$ -subnormal subgroup of  $D$  is normal in  $G$ . If  $G$  is a  $\sigma$ -full group of Sylow type, then  $G$  is a  $P\sigma T$ -group.*

**Corollary 1.8** (See Theorem A in [1]). *Let  $G$  have a normal  $\sigma$ -Hall subgroup  $D$  such that: (i)  $G/D$  is  $\sigma$ -nilpotent, and (ii) every subgroup of  $D$  is normal in  $G$ . Then  $G$  is a  $P\sigma T$ -group.*

In the case when  $\sigma = \sigma^0$ , we get from Theorem B the following

**Corollary 1.9** (Agrawal [10, Theorem 2.4]). *Let  $G$  have a normal Hall subgroup  $D$  such that: (i)  $G/D$  is a  $PST$ -group, and (ii) every subnormal subgroup of  $D$  is normal in  $G$ . Then  $G$  is a  $PST$ -group.*

Some other applications of Theorems A and B and some other characterizations of  $\sigma$ -soluble  $P\sigma T$ -groups we discuss in Section 4.

## 2 Some preliminary results

In view of Theorem B in [4], the following fact is true.

**Lemma 2.1.** *If  $G$  is  $\sigma$ -soluble, then  $G$  is a  $\sigma$ -full group of Sylow type.*

**Lemma 2.2** (See Corollary 2.4 and Lemma 2.5 in [1]). *The class of all  $\sigma$ -nilpotent groups  $\mathfrak{N}_\sigma$  is closed under taking products of normal subgroups, homomorphic images and subgroups. Moreover, if  $E$  is a normal subgroup of  $G$  and  $E/E \cap \Phi(G)$  is  $\sigma$ -nilpotent, then  $E$  is  $\sigma$ -nilpotent.*

**Lemma 2.3** (See Proposition 2.2.8 in [23]). *If  $N$  is a normal subgroup of  $G$ , then  $(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma} N/N$ .*

**Lemma 2.4** (See Knyagina and Monakhov [24]). *Let  $H$ ,  $K$  and  $N$  be pairwise permutable subgroups of  $G$  and  $H$  be a Hall subgroup of  $G$ . Then  $N \cap HK = (N \cap H)(N \cap K)$ .*

**Lemma 2.5** (See Lemma 2.8 in [1]). *Let  $A$ ,  $K$  and  $N$  be subgroups of  $G$ . Suppose that  $A$  is  $\sigma$ -subnormal in  $G$  and  $N$  is normal in  $G$ .*

(1) *If  $N \leq K$  and  $K/N$  is  $\sigma$ -subnormal in  $G/N$ , then  $K$  is  $\sigma$ -subnormal in  $G$ .*

(2)  *$A \cap K$  is  $\sigma$ -subnormal in  $K$ .*

(3) *If  $A$  is a  $\sigma$ -Hall subgroup of  $G$ , then  $A$  is normal in  $G$ .*

(4) *If  $H \neq 1$  is a Hall  $\Pi$ -subgroup of  $G$  and  $A$  is not a  $\Pi'$ -group, then  $A \cap H \neq 1$  is a Hall  $\Pi$ -subgroup of  $A$ .*

(5)  $AN/N$  is  $\sigma$ -subnormal in  $G/N$ .

(6) If  $K$  is a  $\sigma$ -subnormal subgroup of  $A$ , then  $K$  is  $\sigma$ -subnormal in  $G$ .

**Lemma 2.6** (See Lemmas 2.8, 3.1 and Theorem B in [1]). *Let  $H$ ,  $K$  and  $R$  be subgroups of  $G$ . Suppose that  $H$  is  $\sigma$ -permutable in  $G$  and  $R$  is normal in  $G$ . Then:*

(1)  $H$  is  $\sigma$ -subnormal in  $G$ .

(2) The subgroup  $HR/R$  is  $\sigma$ -permutable in  $G/R$ .

(3) If  $K$  is a  $\sigma_i$ -group, then  $K$  is  $\sigma$ -permutable in  $G$  if and only if  $O^{\sigma_i}(G) \leq N_G(K)$ .

(4) If  $G$  is a  $\sigma$ -full group of Sylow type and  $H \leq K$ , then  $H$  is  $\sigma$ -permutable in  $K$ .

(5) If  $G$  is a  $\sigma$ -full group of Sylow type,  $R \leq K$  and  $K/R$  is  $\sigma$ -permutable in  $G/R$ , then  $K$  is  $\sigma$ -permutable in  $G$ .

(6)  $H/H_G$  is  $\sigma$ -nilpotent.

**Lemma 2.7.** *The following statements hold:*

(i)  $G$  is a  $P\sigma T$ -group if and only if every  $\sigma$ -subnormal subgroup of  $G$  is  $\sigma$ -permutable in  $G$ .

(ii) If  $G$  is a  $P\sigma T$ -group, then every quotient  $G/N$  of  $G$  is also a  $P\sigma T$ -group.

**Proof.** (i) This follows from Lemmas 2.5(6) and 2.6(1).

(ii) Let  $H/N$  be a  $\sigma$ -subnormal subgroup of  $G/N$ . Then  $H$  is a  $\sigma$ -subnormal subgroup of  $G$  by Lemma 2.5(1), so  $H$  is  $\sigma$ -permutable in  $G$  by hypothesis and Part (i). Hence  $H/N$  is  $\sigma$ -permutable in  $G/N$  by Lemma 2.6(2). Hence  $G/N$  is a  $P\sigma T$ -group by Part (i).

The lemma is proved.

### 3 Proofs of Theorems A and B

**Proof of Theorem B.** Since  $G$  is a  $\sigma$ -full group of Sylow type by hypothesis, it possesses a complete Hall  $\sigma$ -set  $\mathcal{H} = \{H_1, \dots, H_t\}$  and a subgroup  $H$  of  $G$  is  $\sigma$ -permutable in  $G$  if and only if  $HH_i^x = H_i^x H$  for all  $H_i \in \mathcal{H}$  and  $x \in G$ . We can assume without loss of generality that  $H_i$  is a  $\sigma_i$ -group for all  $i = 1, \dots, t$ .

Assume that this theorem is false and let  $G$  be a counterexample of minimal order. Then  $D \neq 1$  and for some  $\sigma$ -subnormal subgroup  $H$  of  $G$  and for some  $x \in G$  and  $k \in I$  we have  $HH_k^x \neq H_k^x H$  by Lemma 2.7(i). Let  $E = H_k^x$ .

(1) *The hypothesis holds for every quotient  $G/N$  of  $G$ .*

It is clear that  $G/N$  is a  $\sigma$ -full group of Sylow type and  $DN/N$  is a normal  $\sigma$ -Hall subgroup of  $G/N$ . On the other hand,

$$(G/N)/(DN/N) \simeq G/DN \simeq (G/D)/(DN/D),$$

so  $(G/N)/(DN/N)$  is a  $P\sigma T$ -group by Lemma 2.7(ii). Finally, let  $H/N$  be a  $\sigma$ -subnormal subgroup of  $DN/N$ . Then  $H = N(H \cap D)$  and, by Lemma 2.5(1),  $H$  is  $\sigma$ -subnormal in  $G$ . Hence  $H \cap D$  is  $\sigma$ -subnormal in  $D$  by Lemma 2.5(2), so  $H \cap D$  is normal in  $G$  by hypothesis. Thus  $H/N = N(H \cap D)/N$  is normal in  $G/N$ . Therefore the hypothesis holds for  $G/N$ .

(2)  $H_G = 1$ .

Assume that  $H_G \neq 1$ . Clearly,  $H/H_G$  is  $\sigma$ -subnormal in  $G/H_G$ . Claim (1) implies that the hypothesis holds for  $G/H_G$ , so the choice of  $G$  implies that  $G/H_G$  is a  $P\sigma T$ -group. Hence

$$(H/H_G)(EH_G/H_G) = (EH_G/H_G)(H/H_G)$$

by Lemma 2.7(i). Therefore  $HE = EH$ , a contradiction. Hence  $H_G = 1$ .

(3)  $DH = D \times H$ .

By Lemma 2.5(2),  $H \cap D$  is  $\sigma$ -subnormal in  $D$ . Hence  $H \cap D$  is normal in  $G$  by hypothesis, which implies that  $H \cap D = 1$  by Claim (2). Lemma 2.5(2) implies also that  $H$  is  $\sigma$ -subnormal in  $DH$ . But  $H$  is a  $\sigma$ -Hall subgroup of  $DH$  since  $D$  is a  $\sigma$ -Hall subgroup of  $G$  and  $H \cap D = 1$ . Therefore  $H$  is normal in  $DH$  by Lemma 2.5(3), so  $DH = D \times H$ .

*Final contradiction.* Since  $D$  is a  $\sigma$ -Hall subgroup of  $G$ , then either  $E \leq D$  or  $E \cap D = 1$ . But the former case is impossible by Claim (3) since  $HE \neq EH$ , so  $E \cap D = 1$ . Therefore  $E$  is a  $\Pi'$ -subgroup of  $G$ , where  $\Pi = \sigma(D)$ . By the Schur-Zassenhaus theorem,  $D$  has a complement  $M$  in  $G$ . Then  $M$  is a Hall  $\Pi'$ -subgroup of  $G$  and so for some  $x \in G$  we have  $E \leq M^x$  since  $G$  is a  $\sigma$ -full group of Sylow type. On the other hand,  $H \cap M^x$  is a Hall  $\Pi'$ -subgroup of  $H$  by Lemma 2.5(4) and hence  $H \cap M^x = H \leq M^x$ . Lemma 2.5(2) implies that  $H$  is  $\sigma$ -subnormal in  $M^x$ . But  $M^x \simeq G/D$  is a  $P\sigma T$ -group by hypothesis, so  $HE = EH$  by Lemma 2.7(i). This contradiction completes the proof of the theorem.

**Sketch of the proof of Theorem A.** Since  $G$  is  $\sigma$ -soluble by hypothesis,  $G$  is a  $\sigma$ -full group of Sylow type by Lemma 2.1. Let  $\mathcal{H} = \{H_1, \dots, H_n\}$  be a complete Hall  $\sigma$ -set of  $G$ . We can assume without loss of generality that  $H_i$  is a  $\sigma_i$ -group for all  $i = 1, \dots, n$ .

First suppose that  $G$  is a  $P\sigma T$ -group. We show that Conditions (i) and (ii) hold for  $G$  in this case. Assume that this is false and let  $G$  be a counterexample of minimal order. Then  $D \neq 1$ .

(1) *If  $R$  is a non-identity normal subgroup of  $G$ , then Conditions (i) and (ii) hold for  $G/R$*  (Since the hypothesis holds for  $G/R$  by Lemma 2.7(ii), this follows from the choice of  $G$ ).

(2) *If  $E$  is a proper  $\sigma$ -subnormal subgroup of  $G$ , then  $E^{\mathfrak{N}_\sigma} \leq D$  and Conditions (i) and (ii) hold for  $E$ .*

Every  $\sigma$ -subnormal subgroup  $H$  of  $E$  is  $\sigma$ -subnormal in  $G$  by Lemma 2.5(6), so  $H$  is  $\sigma$ -permutable in  $G$  by Lemma 2.7(i). Thus  $H$  is  $\sigma$ -permutable in  $E$  by Lemma 2.6(4). Therefore  $E$  is a  $\sigma$ -soluble  $P\sigma T$ -group by Lemma 2.7(i), so Conditions (i) and (ii) hold for  $E$  by the choice of  $G$ . Moreover, since  $G/D \in \mathfrak{N}_\sigma$  and  $\mathfrak{N}_\sigma$  is a hereditary class by Lemma 2.2,  $E/E \cap D \simeq ED/D \in \mathfrak{N}_\sigma$  and so

$$E^{\mathfrak{N}_\sigma} \leq E \cap D \leq D.$$

(3)  $D$  is nilpotent.

(4)  $D$  is a Hall subgroup of  $G$ . Hence  $D$  has a  $\sigma$ -nilpotent complement  $M$  in  $G$ .

Suppose that this is false and let  $P$  be a Sylow  $p$ -subgroup of  $D$  such that  $1 < P < G_p$ , where  $G_p \in \text{Syl}_p(G)$ . We can assume without loss of generality that  $G_p \leq H_1$ .

( $a^0$ )  $D = P$  is a minimal normal subgroup of  $G$ .

Let  $R$  be a minimal normal subgroup of  $G$  contained in  $D$ . Since  $D$  is nilpotent by Claim (3),  $R$  is a  $q$ -group for some prime  $q$ . Moreover,  $D/R = (G/R)^{\mathfrak{N}_\sigma}$  is a Hall subgroup of  $G/R$  by Claim (1) and Lemma 2.3. Suppose that  $PR/R \neq 1$ . Then  $PR/R \in \text{Syl}_p(G/R)$ . If  $q \neq p$ , then  $P \in \text{Syl}_p(G)$ . This contradicts the fact that  $P < G_p$ . Hence  $q = p$ , so  $R \leq P$  and therefore  $P/R \in \text{Syl}_p(G/R)$  and we again get that  $P \in \text{Syl}_p(G)$ . This contradiction shows that  $PR/R = 1$ , which implies that  $R = P$  is the unique minimal normal subgroup of  $G$  contained in  $D$ . Since  $D$  is nilpotent, a  $p$ -complement  $E$  of  $D$  is characteristic in  $D$  and so it is normal in  $G$ . Hence  $E = 1$ , which implies that  $R = D = P$ .

( $b^0$ )  $D \not\leq \Phi(G)$ . Hence for some maximal subgroup  $M$  of  $G$  we have  $G = D \rtimes M$  (This follows from ( $a^0$ ) and Lemma 2.2 since  $G$  is not  $\sigma$ -nilpotent).

( $c^0$ ) If  $G$  has a minimal normal subgroup  $L \neq D$ , then  $G_p = D \times (L \cap G_p)$ . Hence  $O_{p'}(G) = 1$ .

Indeed,  $DL/L \simeq D$  is a Hall subgroup of  $G/L$  by Claim (1) and lemma 2.3. Hence  $G_p L/L = DL/L$ , so  $G_p = D \times (L \cap G_p)$ . Thus  $O_{p'}(G) = 1$  since  $D < G_p$  by Claim ( $a^0$ ).

( $d^0$ )  $V = C_G(D) \cap M$  is a normal subgroup of  $G$  and  $C_G(D) = D \times V \leq H_1$ .

In view of Claims ( $a^0$ ) and ( $b^0$ ),  $C_G(D) = D \times V$ , where  $V = C_G(D) \cap M$  is a normal subgroup of  $G$ . Moreover,  $V \simeq DV/D$  is  $\sigma$ -nilpotent by Lemma 2.2. Let  $W$  be a  $\sigma_1$ -complement of  $V$ . Then  $W$  is characteristic in  $V$  and so it is normal in  $G$ . Therefore we have ( $d^0$ ) by Claim ( $c^0$ ).

( $e^0$ )  $G_p \neq H_1$ .

Assume that  $G_p = H_1$ . Let  $Z$  be a subgroup of order  $p$  in  $Z(G_p) \cap D$ . Then, since  $O^{\sigma_1}(G) = O^p(G)$ ,  $Z$  is normal in  $G$  by Lemmas 2.6(3) and 2.7(i). Hence  $D = Z < G_p$  by Claim ( $a^0$ ) and so  $D < C_G(D)$ . Then  $V = C_G(D) \cap M \neq 1$  is a normal subgroup of  $G$  and  $V \leq H_1 = G_p$  by Claim ( $d^0$ ). Let  $L$  be a minimal normal subgroup of  $G$  contained in  $V$ . Then  $G_p = D \times L$  is a normal elementary abelian subgroup of  $G$  by Claim ( $c^0$ ). Therefore every subgroup of  $G_p$  is normal in  $G$  by Lemma 2.6(3). Hence  $|D| = |L| = p$ . Let  $D = \langle a \rangle$ ,  $L = \langle b \rangle$  and  $N = \langle ab \rangle$ . Then  $N \not\leq D$ , so in view of the  $G$ -isomorphisms

$$DN/D \simeq N \simeq NL/L = G_p/L = DL/L \simeq D$$

we get that  $G/C_G(D) = G/C_G(N)$  is a  $p$ -group since  $G/D$  is  $\sigma$ -nilpotent by Lemma 2.2. But then Claim ( $d^0$ ) implies that  $G$  is a  $p$ -group. This contradiction shows that we have ( $e^0$ ).

*Final contradiction for (4).* In view of Theorem A in [4],  $G$  has a  $\sigma_1$ -complement  $E$  such that  $EG_p = G_p E$ . Let  $V = (EG_p)^{\mathfrak{N}_\sigma}$ . By Claim ( $e^0$ ),  $EG_p \neq G$ . On the other hand, since  $D \leq EG_p$  by

Claim  $(a^0)$ ,  $EG_p$  is  $\sigma$ -subnormal in  $G$  by Lemma 2.5(1). Therefore Claim (2) implies that  $V$  is a Hall subgroup of  $EG_p$  and  $V \leq D$ , so for a Sylow  $p$ -subgroup  $V_p$  of  $V$  we have  $|V_p| \leq |P| < |G_p|$ . Hence  $V$  is a  $p'$ -group and so  $V \leq C_G(D) \leq H_1 = G_p$  by Claim  $(d^0)$ . Thus  $V = 1$ . Therefore  $EG_p = E \times G_p$  is  $\sigma$ -nilpotent and so  $E \leq C_G(D) \leq H_1$ . Hence  $E = 1$  and so  $D = 1$ , a contradiction. Thus,  $D$  is a Hall subgroup of  $G$ . Hence  $D$  has a complement  $M$  in  $G$  by the Schur-Zassenhaus theorem and  $M \simeq G/D$  is  $\sigma$ -nilpotent by Lemma 2.2.

(5)  $H_i = O_{\sigma_i}(D) \times S$  for each  $\sigma_i \in \sigma(D)$ .

First assume that  $N = O^{\sigma_i}(D) \neq 1$ . Since  $D$  is nilpotent by Claim (3),  $N$  is a  $\sigma'_i$ -group. Moreover,  $G/N$  is a  $P\sigma T$ -group by Lemma 2.7(ii) and so the choice of  $G$  implies that

$$H_i \simeq H_i N/N = (O_{\sigma_i}(D/N)) \times (V/N) = (O_{\sigma_i}(D)N/N) \times (V/N).$$

Since  $D$  is a Hall subgroup of  $H_i$  by Claim (4),  $DN/N$  is a Hall subgroup of  $H_i N/N$  and so  $V/N$  is a Hall subgroup of  $H_i N/N$ . Hence  $V/N$  is characteristic in  $H_i N/N$ . On the other hand, since  $D/N = (G/N)^{\mathfrak{N}\sigma}$  is  $\sigma$ -nilpotent by Lemma 2.2,  $H_i N/N$  is normal in  $G/N$  and so  $V/N$  is normal in  $G/N$ . The subgroup  $N$  has a complement  $S$  in  $V$  by the Schur-Zassenhaus theorem. Thus  $H_i \cap V = H_i \cap NS = S(H_i \cap N) = S$  is normal in  $H_i$ .

Now assume that  $O^{\sigma_i}(D) = 1$ , that is,  $D$  is a  $\sigma_i$ -group. Then  $H_i$  is normal in  $G$ , so all subgroups of  $H_i$  are  $\sigma$ -permutable in  $G$  by Lemmas 2.5(6), 2.7(i) and hypothesis. Since  $D$  is a normal Hall subgroup of  $H_i$ , it has a complement  $S$  in  $H_i$ . Lemma 2.6(3) implies that  $D \leq O^{\sigma_i}(G) \leq N_G(S)$ . Hence  $H_i = D \times S$ .

(6) Every subgroup  $H$  of  $D$  is normal in  $G$ . Hence every element of  $G$  induces a power automorphism in  $D$ .

Since  $D$  is nilpotent by Claim (3), it is enough to consider the case when  $H \leq O_{\sigma_i}(D) = H_i \cap D$  for some  $\sigma_i \in \sigma(D)$ . Claim (5) implies that  $H_i = O_{\sigma_i}(D) \times S$ . It is clear that  $H$  is subnormal in  $G$ , so  $H$  is  $\sigma$ -permutable in  $G$ . Therefore

$$G = H_i O^{\sigma_i}(G) = (O_{\sigma_i}(D) \times S) O^{\sigma_i}(G) = S O^{\sigma_i}(G) \leq N_G(H)$$

by Lemma 2.6(3).

(7) If  $p$  is a prime such that  $(p-1, |G|) = 1$ , then  $p$  does not divide  $|D|$ . Hence the smallest prime in  $\pi(G)$  belongs to  $\pi(|G : D|)$ . In particular,  $|D|$  is odd.

Assume that this is false. Then, by Claim (6),  $D$  has a maximal subgroup  $E$  such that  $|D : E| = p$  and  $E$  is normal in  $G$ . It follows that  $C_G(D/E) = G$  since  $(p-1, |G|) = 1$ . Hence  $G/E = (D/E) \times (ME/E)$ , where  $ME/E \simeq M \simeq G/D$  is  $\sigma$ -nilpotent. Therefore  $G/E$  is  $\sigma$ -nilpotent. But then  $D \leq E$ , a contradiction. Hence we have (7).

(8)  $D$  is abelian.

In view of Claim (6),  $D$  is a Dedekind group. Hence  $D$  is abelian since  $|D|$  is odd by Claim (7).



From Claims (4)–(8) we get that Conditions (i) and (ii) hold for  $G$ .

Now we show that if Conditions (i) and (ii) hold for  $G$ , then  $G$  is a  $P\sigma T$ -group. Assume that this is false and let  $G$  be a counterexample of minimal order. Then  $D \neq 1$  and, by Lemma 2.7(i), for some  $\sigma$ -subnormal subgroup  $H$  of  $G$  and for some  $x \in G$  and  $k \in I$  we have  $HH_k^x \neq H_k^x H$ . Let  $E = H_k^x$ .

(1<sup>0</sup>) If  $N$  is a minimal normal subgroup of  $G$ , then  $G/N$  is a  $P\sigma T$ -group (Since the hypothesis holds for  $G/N$ , this follows from the choice of  $G$ ).

(2<sup>0</sup>) If  $N$  is a minimal normal subgroup of  $G$ , then  $EHN$  is a subgroup of  $G$ . Hence  $E \cap N = 1$ .

Claim (1<sup>0</sup>) implies that  $G/N$  is a  $P\sigma T$ -group. On the other hand,  $EN/N$  is a Hall  $\sigma_k$ -subgroup of  $G/N$  and, by Lemma 2.5(5),  $HN/N$  is a  $\sigma$ -subnormal subgroup of  $G/N$ . Note also that  $G/N$  is  $\sigma$ -soluble, so every two Hall  $\sigma_k$ -subgroups of  $G/N$  are conjugate by Lemma 2.1. Thus,

$$(HN/N)(EN/N) = (EN/N)(HN/N) = EHN/N$$

by Lemma 2.7(i). Hence  $EHN$  is a subgroup of  $G$ . Since  $G$  is  $\sigma$ -soluble,  $N$  is a  $\sigma_j$ -group for some  $j$ . Hence in the case  $E \cap N \neq 1$  we have  $j = k$ , so  $N \leq E$ . But then  $EHN = EH = HE$ , a contradiction. Thus  $E \cap N = 1$ .

(3<sup>0</sup>)  $|\sigma(D)| > 1$ .

Indeed, suppose that  $\sigma(D) = \{\sigma_i\}$ . Then  $H_i/D$  is normal in  $G/D$  since  $G/D \simeq M$  is  $\sigma$ -nilpotent by hypothesis, so  $H_i = D \times S$  is normal in  $G$ . The subgroup  $S$  is also normal in  $G$  since it is characteristic in  $H_i$ . On the other hand, Theorem B and the choice of  $G$  imply that  $S \neq 1$ .

Let  $R$  and  $N$  be minimal normal subgroups of  $G$  such that  $R \leq D$  and  $N \leq S$ . Then  $R$  is a group of order  $p$  for some prime  $p$  and  $N$  is a  $p'$ -group since  $D$  is a Hall subgroup of  $H_i$ . Hence  $R \cap HN \leq O_p(HN) \leq P$ , where  $P$  is a Sylow  $p$ -subgroup of  $H$ , so  $R \cap HN = R \cap H$ . Claim (2<sup>0</sup>) implies that  $EHR$  and  $EHN$  are subgroups of  $G$ . Therefore from Lemma 2.4 and Claim (2<sup>0</sup>) we get that  $R \cap EHN = R \cap E(HN) = (R \cap E)(R \cap HN) = R \cap H$ . Hence

$$\begin{aligned} EHR \cap EHN &= E(HR \cap EHN) = EH(R \cap EHN) \\ &= EH(R \cap H) = EH \end{aligned}$$

is a subgroup of  $G$ . Thus  $HE = EH$ , a contradiction. Hence we have (3<sup>0</sup>).

*Final contradiction.* Since  $|\sigma(D)| > 1$  by Claim (3<sup>0</sup>) and  $D$  is nilpotent,  $G$  has at least two minimal normal subgroups  $R$  and  $N$  such that  $R, N \leq D$  and  $\sigma(R) \neq \sigma(N)$ . Then at least one of the subgroups  $R$  or  $N$ ,  $R$  say, is a  $\sigma_i$ -group for some  $i \neq k$ . Hence  $R \cap HN \leq O_{\sigma_i}(HN) \leq V$ , where  $V$  is a Hall  $\sigma_i$ -subgroup of  $H$ , since  $N$  is a  $\sigma'_i$ -group and  $G$  is a  $\sigma$ -full group of Sylow type. Hence  $R \cap HN = R \cap H$ . Claim (2<sup>0</sup>) implies that  $EHR$  and  $EHN$  are subgroups of  $G$ . Now, arguing similarly as in the proof of (3<sup>0</sup>), one can show that  $EHR \cap EHN = EH = HE$ . This contradiction completes the proof of the fact that  $G$  is a  $P\sigma T$ -group.

The theorem is proved.

## 4 Some other characterizations of $\sigma$ -soluble $P\sigma T$ -groups

Theorem A and Theorem B in [1] are basic in the sense that many other characterizations of  $\sigma$ -soluble  $P\sigma T$ -groups can be obtained by using these two results. As a partial illustration to this, we give in this section our next three characterizations of  $\sigma$ -soluble  $P\sigma T$ -groups.

1. Recall that  $Z_\sigma(G)$  denotes the  $\sigma$ -hypercentre of  $G$  [26], that is, the largest normal subgroup of  $G$  such that for every chief factor  $H/K$  of  $G$  below  $Z_\sigma(G)$  the semidirect product  $(H/K) \rtimes (G/C_G(H/K))$  is  $\sigma$ -primary.

We say, following [6, p. 20], that a subgroup  $H$  of  $G$  is  $\sigma$ -hypercentrally embedded in  $G$ , if  $H/H_G \leq Z_\sigma(G/H_G)$ .

**Theorem 4.1.** *Let  $G$  be  $\sigma$ -soluble. Then  $G$  is a  $P\sigma T$ -group if and only if every  $\sigma$ -subnormal subgroup of  $G$  is  $\sigma$ -hypercentrally embedded in  $G$ .*

**Proof.** Let  $D = G^{\mathfrak{N}_\sigma}$ . First we show that if  $G$  is a  $P\sigma T$ -group, then every  $\sigma$ -subnormal subgroup  $H$  of  $G$  is  $\sigma$ -hypercentrally embedded in  $G$ . Assume that this is false and let  $G$  be a counterexample with  $|G| + |H|$  minimal. Then  $G/H_G$  is a  $\sigma$ -soluble  $P\sigma T$ -group by Lemma 2.7(ii) and  $H/H_G$  is  $\sigma$ -subnormal in  $G/H_G$  by Lemma 2.5(5). Hence the choice of  $G$  implies that  $H_G = 1$ , so  $H$  is  $\sigma$ -nilpotent by Lemma 2.6(6). Therefore every subgroup of  $H$  is  $\sigma$ -subnormal in  $G$  by Proposition 2.3 in [1] and Lemma 2.5(6). Assume that  $H$  possesses two distinct maximal subgroups  $V$  and  $W$ . Then  $V, W \leq Z_\sigma(G)$  by minimality of  $|G| + |H|$  since  $V_G = 1 = W_G$ , which implies that  $H \leq Z_\sigma(G)$ . Hence  $H$  is a cyclic  $p$ -group for some  $p \in \sigma_i$ .

By Theorem A,  $G = D \rtimes M$ , where  $D$  is a Hall subgroup of  $G$ ,  $M$  is  $\sigma$ -nilpotent and every subgroup of  $D$  is normal in  $G$ . Then  $H \cap D = 1$  and so, in view of Lemma 2.1, we can assume without loss of generality that  $H \leq M$ . Lemma 2.7(i) implies that  $H$  is  $\sigma$ -permutable in  $G$ , so

$$H^G = H^{DM} = H^{O^{\sigma_i}(G)M} = H^M \leq M$$

by Lemma 2.6(3). Hence  $H^G \cap D = 1$  and then, from the  $G$ -isomorphism  $H^G D/D \simeq H^G$ , we deduce that  $H \leq H^G \leq Z_\sigma(G)$ . Therefore  $H$  is  $\sigma$ -hypercentrally embedded in  $G$ . This contradiction completes the proof of the necessity of the condition of the theorem.

*Sufficiency.* It is enough to show that if a  $\sigma$ -subnormal subgroup  $H$  of a  $\sigma$ -soluble group  $G$  is  $\sigma$ -hypercentrally embedded in  $G$ , then  $H$  is  $\sigma$ -permutable in  $G$ . Assume that this is false and let  $G$  be a counterexample with  $|G| + |H|$  minimal. Since  $G$  is  $\sigma$ -soluble, it is a  $\sigma$ -full group of Sylow type by Lemma 2.1. Therefore, in view of Lemma 2.6(5),  $H_G = 1$  and so  $H \leq Z_\sigma(G)$ . It is clear that  $Z_\sigma(G)$  is  $\sigma$ -nilpotent, so  $H = H_1 \times \cdots \times H_t$  for some  $\sigma$ -primary groups  $H_1, \dots, H_t$ . Moreover, the minimality of  $|G| + |H|$  implies that  $H = H_1$  is a  $\sigma_i$ -group for some  $i$ . Hence  $H \leq N$ , where  $N$  is a Hall  $\sigma_i$ -subgroup of  $Z_\sigma(G)$ . Since  $Z_\sigma(G)$  is  $\sigma$ -nilpotent,  $N$  is characteristic in  $Z_\sigma(G)$  and so  $N$  is normal in  $G$ .

Let  $1 = Z_0 < Z_1 < \cdots < Z_t = N$  be a chief series of  $G$  below  $N$  and  $C_i = C_G(Z_i/Z_{i-1})$ . Let

$C = C_1 \cap \cdots \cap C_t$ . Then  $G/C$  is a  $\sigma_i$ -group. On the other hand,  $C/C_G(N) \simeq A \leq \text{Aut}(N)$  stabilizes the series  $1 = Z_0 < Z_1 < \cdots < Z_t = N$ , so  $C/C_G(N)$  is a  $\pi(N)$ -group by [25, Ch. A, 12.4(a)]. Hence  $G/C_G(N)$  is a  $\sigma_i$ -group and so  $O^{\sigma_i}(G) \leq C_G(N)$ . But then  $O^{\sigma_i}(G) \leq C_G(H)$ , so  $H$  is  $\sigma$ -permutable in  $G$  by Lemma 2.6(3). This contradiction completes the proof of the sufficiency of the condition of the theorem.

The theorem is proved.

In the case when  $\sigma = \sigma^0$ , we have  $Z_\sigma(G) = Z_\infty(G)$ . Hence from Theorem 4.1 we get

**Corollary 4.2** (See Theorem 2.4.4 in [6]). *Let  $G$  be soluble. Then  $G$  is a PST-group if and only if every subnormal subgroup  $H$  of  $G$  is hypercentrally embedded in  $G$  (that is,  $H/H_G \leq Z_\infty(G/H_G)$ ).*

2. We say, following [6, p. 68], that  $G$  satisfies property  $\mathcal{Y}_{\sigma_i}$  if whenever  $H \leq K$  are two  $\sigma_i$ -subgroups of  $G$ ,  $H$  is  $\sigma$ -permutable in  $N_G(K)$ .

The idea of the next theorem goes back to the paper [15].

**Theorem 4.3.** *Let  $G$  be  $\sigma$ -soluble. Then  $G$  is a  $P\sigma T$ -group if and only if  $G$  satisfies  $\mathcal{Y}_{\sigma_i}$  for all primes  $i$ .*

**Lemma 4.4.** *Let  $K \leq H$  and  $N$  be subgroups of  $G$ . Suppose that  $K$  is  $\sigma$ -permutable in  $H$  and  $N$  is normal in  $G$ . Then  $KN/N$  is  $\sigma$ -permutable in  $HN/N$ .*

**Proof.** Let  $f : H/H \cap N \rightarrow HN/N$  be the canonical isomorphism from  $H/H \cap N$  onto  $HN/N$ . Then  $f(K(H \cap N)/(H \cap N)) = KN/N$ , so  $KN/N$  is  $\sigma$ -permutable in  $HN/N$  by Lemma 2.6(2).

The lemma is proved.

**Sketch of the proof of Theorem 4.3.** *Necessity.* Let  $H \leq K$  be two  $\sigma_i$ -subgroups of  $G$  and  $N = N_G(K)$ . Then  $H$  is  $\sigma$ -subnormal in  $N$  by Lemma 2.5(6). On the other hand, Corollary 1.2 implies that  $N$  is a  $\sigma$ -soluble  $P\sigma T$ -group. Therefore  $H$  is  $\sigma$ -permutable in  $N$  by Lemma 2.7(i).

*Sufficiency.* It is enough to show that Conditions (i) and (ii) of Theorem A hold for  $G$ . Assume that this is false and let  $G$  be a counterexample of minimal order. Since  $G$  is  $\sigma$ -soluble, it is a  $\sigma$ -full group of Sylow type by Lemma 2.1. Let  $D = G^{\mathfrak{N}_\sigma}$ .

(1) *Every proper subgroup  $E$  of  $G$  is a  $\sigma$ -soluble  $P\sigma T$ -group and  $E^{\mathfrak{N}_\sigma} \leq D$  (This follows from Lemmas 2.2, 2.3, 2.6(4) and the choice of  $G$ ).*

(2)  *$G/N$  is a  $\sigma$ -soluble  $P\sigma T$ -group for every minimal normal subgroup  $N$  of  $G$ .*

Let  $H/N \leq K/N$  be two  $\sigma_i$ -subgroups of  $G/N$ . Since  $G$  is  $\sigma$ -soluble,  $N$  is a  $\sigma_j$ -subgroup for some  $j$ . Assume that  $j \neq i$ . Then there are a Hall  $\sigma_i$ -subgroup  $V$  of  $H$  and a Hall  $\sigma_i$ -subgroup  $W$  of  $K$  such that  $V \leq W$  since  $G$  is a  $\sigma$ -full group of Sylow type. Then  $V$  is  $\sigma$ -permutable in  $N_G(W)$  by hypothesis, so  $H/N = VN/N$  is  $\sigma$ -permutable in  $N_G(W)N/N = N_{G/N}(WN/N) = N_{G/N}(K/N)$  by Lemma 4.4. Similarly we get that  $H/N$  is  $\sigma$ -permutable in  $N_{G/N}(K/N)$  in the case when  $j = i$ .

(3)  *$D$  is nilpotent.*

(4)  $D$  is a Hall subgroup of  $G$  and  $H_i = O_{\sigma_i}(D) \times S$  for each  $\sigma_i \in \sigma(D)$  (See Claims (4) and (5) in the proof of Theorem A and use Claims (1), (2) and (3)).

(5) Every subgroup  $H$  of  $D$  is normal in  $G$ . Hence every element of  $G$  induces a power automorphism in  $D$ .

Since  $D$  is nilpotent by Claim (3), it is enough to consider the case when  $H \leq O_{\sigma_i}(D) = H_i \cap D$  for some  $\sigma_i \in \sigma(D)$ . Hence  $H$  is  $\sigma$ -permutable in  $G$  by hypothesis. Claim (4) implies that  $H_i = O_{\sigma_i}(D) \times S$ . Therefore  $G = H_i O^{\sigma_i}(G) = S O^{\sigma_i}(G) \leq N_G(H)$  by Lemma 2.6(3).

(6)  $D$  is abelian of odd order (See Claims (7) and (8) in the proof of Theorem A and use Claim (5)).

The theorem is proved.

**Corollary 4.5** (Ballester-Bolínches and Esteban-Romero [15], see also Theorem 2.2.9 in [6]).  $G$  is a soluble PST-group if and only if  $G$  satisfies  $\mathcal{Y}_p$  for all primes  $p$ .

**Proof.** It is enough to note that, as it was remarked at the beginning of the proof of Theorem 2.2.9 in [6], every group which satisfies  $\mathcal{Y}_p$  for all primes  $p$  is soluble.

3. We say that a subgroup  $A$  of  $G$  is  $\sigma$ -modular ( $S$ -modular in the case  $\sigma = \sigma^0$ ) provided  $G$  possesses a complete Hall  $\sigma$ -set and  $\langle A, H \cap C \rangle = \langle A, H \rangle \cap C$  for every Hall  $\sigma_i$ -subgroup  $H$  of  $G$  and all  $i \in I$  and  $A \leq C \leq G$ .

**Theorem 4.6.** Let  $G$  be  $\sigma$ -soluble. Then  $G$  is a  $P\sigma T$ -group if and only if every  $\sigma$ -subnormal subgroup  $A$  of  $G$  is  $\sigma$ -modular in every subgroup of  $G$  containing  $A$ .

**Proof.** Since  $G$  is  $\sigma$ -soluble,  $G$  is a  $\sigma$ -full group of Sylow type by Lemma 2.1. Hence  $G$  possesses a complete Hall  $\sigma$ -set  $\mathcal{H} = \{H_1, \dots, H_t\}$  and, for each  $\sigma_i \in \sigma(G)$ , a subgroup  $H$  of  $G$  is a Hall  $\sigma_i$ -subgroup of  $G$  if and only if  $H = H_k^x$  for some  $x \in G$  and  $H_k \in \mathcal{H}$ .

*Sufficiency.* Assume that this is false and let  $G$  be a counterexample of minimal order. Then, in view of Lemma 2.7(i),  $G$  has a  $\sigma$ -subnormal subgroup  $A$  which is not  $\sigma$ -permutable in  $G$ . Hence, for some  $H_i \in \mathcal{H}$  and  $x \in G$ , we have  $AH_i^x \neq H_i^x A$ . Note also that every proper  $\sigma$ -subnormal subgroup  $E$  of  $G$  is a  $P\sigma T$ -group. Indeed,  $E$  is clearly  $\sigma$ -soluble and if  $H$  is a  $\sigma$ -subnormal subgroup of  $E$ , then  $H$  is  $\sigma$ -subnormal in  $G$  by Lemma 2.5(6). Hence  $H$  is  $\sigma$ -modular in every subgroup of  $E$  containing  $H$  by hypothesis. Thus the hypothesis holds for  $E$  and so  $E$  is a  $P\sigma T$ -group by the choice of  $G$ .

By definition, there is a subgroup chain  $A = A_0 \leq A_1 \leq \dots \leq A_n = G$  such that either  $A_{i-1} \trianglelefteq A_i$  or  $A_i/(A_{i-1})_{A_i}$  is  $\sigma$ -primary for all  $i = 1, \dots, n$ . We can assume without loss of generality that  $M = A_{n-1} < G$ . Then  $M$  is a  $P\sigma T$ -group since  $M$  is clearly  $\sigma$ -subnormal in  $G$ , so  $A$  is  $\sigma$ -permutable in  $M$  by Lemma 2.7(i). Moreover, the  $\sigma$ -modularity of  $A$  in  $G$  implies that

$$M = M \cap \langle A, H_i^x \rangle = \langle A, (M \cap H_i^x) \rangle.$$

On the other hand, by Lemma 2.5(4),  $M \cap H_i^x$  is a Hall  $\sigma_i$ -subgroup of  $M$ , where  $\{\sigma_i\} = \sigma(H_i)$ . Hence  $M = A(M \cap H_i^x) = (M \cap H_i^x)A$ . If  $H_i^x \leq M_G$ , then  $A(M \cap H_i^x) = AH_i^x = H_i^x A$  and so

$$H_i^x \not\leq M_G.$$

Now note that  $H_i^x M = M H_i^x$ . Indeed, if  $M$  is normal in  $G$ , it is clear. Otherwise,  $G/M_G$  is  $\sigma$ -primary and so  $G = M H_i^x = H_i^x M$  since  $H_i^x \not\leq M_G$  and  $H_i \in \mathcal{H}$ . Therefore

$$H_i^x A = H_i^x (M \cap H_i^x) A = H_i^x M = M H_i^x = H_i^x (M \cap H_i^x) A = H_i^x A.$$

This contradiction completes the proof of the sufficiency of the condition of the theorem.

*Necessity.* In view of Lemma 2.6(4), it is enough to show that if  $A$  is a  $\sigma$ -subnormal subgroup of  $G$ , then  $A$  is  $\sigma$ -modular in  $G$ . First note that  $A$  is  $\sigma$ -permutable in  $G$  by Lemma 2.7(i). Therefore for every subgroup  $C$  of  $G$  containing  $A$ , for every  $i \in I$ , and for all Hall  $\sigma_i$ -subgroup  $H$  of  $G$  we have

$$\langle A, H \cap C \rangle = A(H \cap C) = AH \cap C = \langle A, H \rangle \cap C,$$

so  $A$  is  $\sigma$ -modular in  $G$ .

The theorem is proved.

From Theorem 4.6 we get the following characterization of soluble *PST*-groups.

**Corollary 4.7.** *Let  $G$  be soluble. Then  $G$  is a *PST*-group if and only if every subnormal subgroup  $A$  of  $G$  is  $S$ -modular in every subgroup of  $G$  containing  $A$ .*

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